EEM 342: Fundamentals of Control Systems  
2002–2003 Spring Semester  
Solutions to Homework # 1

1. a. i) It is static, since output at time $t$ depends only on input at time $t$, not at other times.
   ii) It is causal, since output at time $t$ does not depend on input at times greater than $t$. In fact, static systems are always causal.
   iii) It is continuous-time, since the input and the output are defined (implicitly) for all $t \in \mathcal{T}$, which is a continuous set.
   iv) It is time-invariant, since any time-shift of the input produces the same time-shift in the output. i.e., if $u(t) = u^1(t)$ produces $y(t) = y^1(t) = \sqrt{(u^1(t))^2}$, then $u(t) = u^1(t - T)$ produces $y(t) = \sqrt{(u^1(t - T))^2} = y^1(t - T)$, for any input $u^1$, and any $T$.
   v) Since we take the positive square root, the given relation can be written as $y(t) = |u(t)|$. Now, multiply the input by $-1$ (or by any negative number), the output stays the same (or is multiplied by the absolute value of the negative number, not by itself). Thus, input-output relation does not satisfy homogeneity (in fact, it does not satisfy superposition either). Therefore, the system is not linear.

b. i–iii) It is static, causal, and continuous-time, as for case (a).
   iv) It is time-varying, since a time-shift of the input does not normally produce the same time-shift in the output. e.g., $u(t) = u^1(t)$ produces $y(t) = y^1(t) = tu^1(t)$, but $u(t) = u^1(t - T)$ produces $y(t) = tu^1(t - T) \neq y^1(t - T)$, in general.
   v) It is linear, since the input-output relation satisfy linearity: If $y^1(t) = tu^1(t)$ and $y^2(t) = tu^2(t)$, then $[a_1u^1(t) + a_2u^2(t)] = a_1(tu^1(t)) + a_2(tu^2(t)) = a_1y^1(t) + a_2y^2(t)$, for any input signals $u^1$ and $u^2$ and any constants $a_1$ and $a_2$.

c. i) It is dynamic, since output at time $t$ depends on input at times other than $t$.
   ii) It is non-causal, since output at time $t$ does depend on input at times greater than $t$.
   iii) It is continuous-time, since the input and the output are defined (implicitly) for all $t \in \mathcal{T}$.
   iv) It is time-invariant, since $\int_{t-1}^{t+1} u(\tau - T)d\tau = \int_{(t-T)-1}^{(t-T)+1} u(\nu)d\nu$, for any $u$ and for any $T$, thus any time-shift of the input produces the same time-shift in the output.
   v) It is linear, since $\int_{t-1}^{t+1} (a_1u^1(\tau) + a_2u^2(\tau))d\tau = a_1\left(\int_{t-1}^{t+1} u^1(\tau)d\tau\right) + a_2\left(\int_{t-1}^{t+1} u^2(\tau)d\tau\right)$, for any input signals $u^1$ and $u^2$ and for any constants $a_1$ and $a_2$, thus the input-output relation satisfies linearity.
d. i) It is dynamic, since output at time $t$ depends on input at times other than $t$.
ii) It is causal, since output at time $t$ does not depend on input at times greater than $t$.
iii) It is continuous-time, since the input and the output are defined (implicitly) for all $t \in \mathcal{T}$.
iv) It is time-varying, since a time shift in the input does not, in general produce the same time shift in the output: 
\[
\int_{-\infty}^{t} \sin(\tau) u(\tau - T) d\tau = \int_{-\infty}^{t} \sin(\nu + T) u(\nu) d\nu \neq \int_{-\infty}^{t} \sin(\nu) u(\nu) d\nu.
\]
v) It is linear, since 
\[
\int_{-\infty}^{t} \sin(\tau) \left( a_1 u^1(\tau) + a_2 u^2(\tau) \right) d\tau = a_1 \left( \int_{-\infty}^{t} \sin(\tau) u^1(\tau) d\tau \right) + a_2 \left( \int_{-\infty}^{t} \sin(\tau) u^2(\tau) d\tau \right),
\]
for any input signals $u^1$ and $u^2$ and for any constants $a_1$ and $a_2$.

e. i) It is dynamic, since output at time $k$ depends on input at times other than $k$.
ii) It is non-causal, since output at time $k$ depends on the input at time $k+1$.
iii) It is discrete-time, since the input and the output are defined only at discrete time instants, $k \in \mathcal{K}$.
iv) It is time-invariant, since 
\[
\sum_{l=-\infty}^{k+1} u(l - m) = \sum_{n=-\infty}^{k-m+1} u(n),
\]
for any input sequence $u$ and for any integer $m$, thus any time-shift of the input produces the same time-shift in the output.
v) It is linear, since 
\[
\sum_{l=-\infty}^{k+1} \left( a_1 u^1(l) + a_2 u^2(l) \right) = a_1 \left( \sum_{l=-\infty}^{k+1} u^1(l) \right) + a_2 \left( \sum_{l=-\infty}^{k+1} u^2(l) \right),
\]
for any input sequences $u^1$ and $u^2$ and for any constants $a_1$ and $a_2$, thus the input-output relation satisfies linearity.

f. i) It is dynamic, since output at time $k$ depends on input at times other than $k$. Note that the system can also be defined as: 
\[
y(k) = u(k-1) - y(k-1) = \sum_{l=-\infty}^{k-1} (-1)^{l-k+1} u(l).
\]
ii) It is causal, since output at time $k$ does not depend on input at times greater than $k$.
iii) It is discrete-time, since the input and the output are defined only at discrete time instants, $k \in \mathcal{K}$.
iv) It is time-invariant, since the difference equation is constant coefficient. This makes the output to shift in time when the input is shifted in time. Also note that, 
\[
\sum_{l=-\infty}^{k-1} (-1)^{l-k+1} u(l - m) = \sum_{n=-\infty}^{k-m-1} (-1)^{n+m-k+1} u(n) = \sum_{n=-\infty}^{(k-m)-1} (-1)^{n-(k-m)+1} u(n).
\]
v) It is linear, since it is described by a linear difference equation. You can verify that the input-output relation satisfies linearity.
g. i) It is dynamic, since output at time \( t \) depends on the time derivative of the input at time \( t \) (one needs to know \( u(t) \) either just before \( t \) or just after \( t \) in order to find the derivative at time \( t \)).

ii) It is causal if we calculate the derivative of the input by using only the past values of the input. Otherwise, it is non-causal.

iii) It is continuous-time, since the input and the output are defined (implicitly) for all \( t \in \mathcal{T} \).

iv) It is time-invariant, since when \( u \) is shifted in time, its derivative also shift the same amount in time.

ev) It is linear, since 
\[
\frac{d}{dt} \left( a_1 u_1(t) + a_2 u_2(t) \right) = a_1 \frac{d}{dt} u_1(t) + a_2 \frac{d}{dt} u_2(t),
\]
for all input signals \( u_1 \) and \( u_2 \) and for all constants \( a_1 \) and \( a_2 \).

h. i) It is dynamic, since the input-output relation is defined by a differential equation, whose solution, \( y(t) \), depends on either past values of the forcing function, \( u(t) \), (if the differential equation is solved forward in time) or future values of it (if the differential equation is solved backward in time).

ii) It is causal if the differential equation is solved forward in time. If the differential equation is solved backward in time, then the system is anti-causal (thus, non-causal). We, however, implicitly assume that such equations are solved forward in time, thus systems described by such differential equations are assumed to be causal.

iii) It is continuous-time, since the input and the output are defined (implicitly) for all \( t \in \mathcal{T} \). In fact, only continuous-time systems can be described by differential equations.

iv) It is time-invariant, since it is described by a constant coefficient differential equation. Note that, if \( y(t) = y^1(t) \) is the solution of such a differential equation with \( u(t) = u^1(t) \), then \( y(t) = y^1(t-T) \) is the solution when \( u(t) = u^1(t-T) \), for all forcing functions \( u^1 \) and for all \( T \).

ev) It is linear, since it is described by a linear differential equation. Note that, if \( y(t) = y^1(t) \) is the solution of such a differential equation with \( u(t) = u^1(t) \) and \( y(t) = y^2(t) \) is the solution when \( u(t) = u^2(t) \), then \( y(t) = a_1 y^1(t) + a_2 y^2(t) \) is the solution when \( u(t) = a_1 u^1(t) + a_2 u^2(t) \), for all forcing functions \( u^1 \) and \( u^2 \) and for all constants \( a_1 \) and \( a_2 \).

j. i-iii) As for case (h), it is dynamic, causal (assuming that the differential equation is solved forward in time), and continuous-time.

iv) It is time-varying, since the given differential equation is not constant coefficient, due to the term \( e^{-t} \), e.g., solve the differential equation with \( u(t) = 1(t) \): the unit step function. Then, solve it for \( u(t) = 1(t-1) \). You will note that the solution in the latter case is not a shifted version of the solution in the former case.

v) It is non-linear, since the given differential equation is not linear, due to the cross term \( u(t) y(t) \) and the term \( e^{-t} \), e.g., solve the differential equation with \( u(t) = 1(t) \). Then, solve it for \( u(t) = 2 \times 1(t) \). You will note that the solution in the latter case is not two times the solution in the former case.
k. i,iii) As for case (f), it is dynamic and discrete-time.

ii) Note that, \( y(k) = (k - 1)^2 u(k + 1) - y(k - 1) = \sum_{l=-\infty}^{k+1} (-1)^{l-k-1}(l-2)^2 u(l) \); i.e., the output at time \( k \) depends on input at time \( k + 1 \); thus the system is non-causal.

iv) It is time-varying, since it is described by a non-constant coefficient difference equation.

Note that, if \( y(k) = y^1(k) \) is the solution of such a difference equation with \( u(k) = u^1(k) \), then \( y(k) = y^1(k - m) \) is, in general, not the solution when \( u(k) = u^1(k - m) \).

v) It is linear, since it is described by a linear difference equation. Note that, if \( y(k) = y^1(k) \) is the solution of such a difference equation with \( u(k) = u^1(k) \) and \( y(k) = y^2(k) \) is the solution when \( u(k) = u^2(k) \), then \( y(k) = a_1 y^1(k) + a_2 y^2(k) \) is the solution when \( u(k) = a_1 u^1(k) + a_2 u^2(k) \), for all forcing functions \( u^1 \) and \( u^2 \) and for all constants \( a_1 \) and \( a_2 \).

2. The system described is a closed-loop control system, since the variable to be controlled (the speed of the car) is measured and feedback to calculate the control signal. Here, we can think of the car as the plant and the engine (together with the throttle) as the actuator. Alternatively, engine and the car can be taken as the plant and the throttle can be taken as the actuator. In both cases, the speedometer is the sensor and the microprocessor is the controller. The car’s speed is the plant output (y), the speedometer signal is the measurement (m), the desired speed setting is the reference (r), throttle setting signal is the control signal (c), anything (other than the plant input) that affect car’s dynamics (such as road and whether conditions) is a disturbance (d). In the first alternative (where we take the engine as a part of the actuator), the torque produced by the engine to drive the car’s wheels is the plant input (u). If we take the engine as a part of the plant, however, the fuel flow into the engine, adjusted by the throttle (the actuator), is the plant input (u). In this latter case, the torque produced by the engine is an internal variable to the plant. A block diagram of the system is as follows: