1. Consider the system described by the following state equations:

\[
\begin{align*}
\dot{x}(t) &= \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} u(t) \\
y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 & 2 \end{bmatrix} u(t)
\end{align*}
\]

a) (6 points) Find the transfer function matrix of the system.

b) (12 points) Let \( x(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \) and \( u(t) = \begin{bmatrix} 1 \\ e^{-t} \end{bmatrix} \) for \( t \geq 0 \). Find \( x(t) \) and \( y(t) \) for \( t \geq 0 \).

c) (12 points) Determine whether the system is (i) bounded-input bounded-output stable, (ii) stable in the sense of Lyapunov, and (iii) asymptotically stable.

2. (50 points) Consider linear time-invariant (LTI) continuous-time plants with transfer function

\[
\begin{align*}
a) \quad G(s) &= \frac{s + 1}{s - 1} \\
b) \quad G(s) &= \frac{s}{s - 1}
\end{align*}
\]

For each plant, determine whether it is possible to design a LTI discrete-time controller to be implemented on a digital computer, such that (i) the closed loop system is internally stable, (ii) the sampled output of the plant can track constant reference signals with no steady-state error, and (iii) the sampled output settles to 5% of its steady state value within 3.5 seconds in response to a step reference input. Assume that the D/A includes a zero order hold and the sampling period for the D/A and A/D (which are assumed to be synchronized) is \( T = 693 \) milliseconds. If your answer is positive, design such a controller (write the transfer function of the controller and draw a block diagram of the overall system). If your answer is negative, explain the reason. **Hint:** \( \ln(2) = 0.693 \).

3. Consider the LTI discrete-time system which has the transfer function matrix

\[
G(z) = \begin{bmatrix}
\frac{z + 1}{z^2 - 1} \\
\frac{z^2 + 1}{z^2 - z}
\end{bmatrix}
\]

a) (10 points) Obtain the state equations of a minimal realization of this transfer function matrix.

b) (10 points) Write a computer program to implement the above realization on a digital computer.
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1. a) \[(sI - A)^{-1} = \begin{bmatrix} s + 1 & 0 \\ -1 & s \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{s+1} & 0 \\ \frac{1}{s+1} & \frac{1}{s} \end{bmatrix}\]
\[G(s) = C(sI - A)^{-1}B + D = \begin{bmatrix} \frac{1}{s+1} \\ \frac{1}{s+1} \end{bmatrix}^2 \begin{bmatrix} 2 \end{bmatrix}\]

b) \[e^{At} = L^{-1}[(sI - A)^{-1}] = \begin{bmatrix} e^{-t} & 0 \\ 1 - e^{-t} & 1 \end{bmatrix}\]
\[x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau = \begin{bmatrix} 1 \\ 3 - e^{-t} + t \end{bmatrix}, \quad t \geq 0\]
\[y(t) = Cx(t) + Du(t) = 2 + 2e^{-t}, \quad t \geq 0\]

c) i) The system transfer function matrix has only one pole at \(-1\), which is in the OLHP. Therefore, the system is BIBO stable.

ii) The dynamics matrix of the system has one eigenvalue at \(-1\), which is in the OLHP, and one simple (thus having index 1) eigenvalue at the origin. Therefore, the system is stable in the sense of Lyapunov.

iii) The system is not asymptotically stable, due to the eigenvalue at the origin.

2. a) \(G(s)\) does not have any zeros at the origin. Therefore, it is possible to design a controller to satisfy (i)–(iii). For this, the closed-loop system must have the following structure:
which has the discrete-time equivalent:

\[
G_d(z) = \left(1 - \frac{1}{z}\right) Z \left[\frac{s + 1}{s(s - 1)}\right] = \frac{z}{z - 2}
\]

To satisfy (ii), the open-loop system (with t.f. \( G_d(z)C(z) \)) must have a pole at \( z = 1 \). Since \( G_d(z) \) does not have such a pole, we include this in \( C(z) \) and let

\[
C(z) = K \frac{z - a}{z - 1}
\]

To satisfy (i), the closed-loop poles must be inside the unit circle. To satisfy (iii), we choose the closed-loop poles inside the circle with radius \( \rho = (0.05)^{1/2} \), where \( k_i = \begin{bmatrix} \rho \\ \frac{3.5}{0.603} \end{bmatrix} = 5 \) and \( m = 2 \) (number of finite zeros). Thus, \( \rho = 0.37 \).

To achieve above, we let \( a = 0.4 \). Then, the loci of the closed-loop poles, as \( K \) is changed from 0 to \( \infty \) is as follows:

At the break-in point \( z = z_1 \) we have

\[
\frac{d}{dz} \left[ \frac{z(z - 0.4)}{(z - 2)(z - 1)} \right]_{z = z_1} = 0
\]

From which we find \( z_1 = 0.24 \). Then

\[
K_1 = \frac{(1 - z_1)(2 - z_1)}{z_1(0.4 - z_1)} = 34.8
\]

We choose \( K = 35 \), which is slightly larger than \( K_1 \), so that one of the closed-loop poles will be slightly to the left of \( z_1 \), and the other slightly to the right of \( z_1 \). Thus we obtain

\[
C(z) = \frac{35z - 14}{z - 1}
\]

It can be checked that, with this controller, the closed-loop system, whose block diagram is shown on the previous page, satisfies requirements (i)-(iii).
b) Since $G(s)$ has a zero at the origin, it is not possible to design a (discrete- or continuous-time) controller to satisfy (i) and (ii) simultaneously. Note that, with this $G(s)$, $G_d(z)$ will have a zero at $z = 1$. In general, the discrete-time equivalent of a type $N$ continuous-time system, preceded by a sampler and a ZOH, is also type $N$.

3.

$$G(z) = \begin{bmatrix} \frac{z + 1}{z^2 - 1} \\ \frac{z^2 + 1}{z^2 - z} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} z \\ z + 1 \end{bmatrix} \frac{1}{z^2 - z}$$

a) A controllable canonical form realization can be obtained by noting that:

\[ x(k+1) = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k) \]

\[ y(k) = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k) \]

b) The following code can be used to implement the above realization:

```plaintext
x1=0; x2=0;
WHILE 0==0,
   READ u
   y1 = x2;
   y2 = x1 + x2 + u;
   WRITE y1, y2
   x1n = x2;
   x2n = x2 + u;
   x1=x1n; x2=x2n;
   WAIT FOR NEXT SAMPLING INSTANT
END
```