0. (1 point) Write the group you are registered to on the top-right corner of your answer sheet.

1. (59 points) Consider the linear time-invariant (LTI) plants with transfer function

   a) \( G(s) = \frac{s}{s - 2} \)
   b) \( G(s) = \frac{2}{s - 2} \)

   where the unit of time is seconds and the sensor(s) and the actuator(s) are assumed to be ideal with unit gain. It is also assumed that neither plant has any hidden unstable modes.

   For each plant, determine whether it is possible to design a LTI proper controller such that

   (i) the closed loop system is internally stable,
   (ii) the output of the plant can track constant reference signals with no steady-state error in the presence of sinusoidal additive disturbances of frequency 0.5 Hz,
   (iii) the output settles to its steady-state value at a rate no slower than that of \( e^{-t} \), and
   (iv) the absolute value of the steady-state error in response to a ramp reference signal is no more than 10% of the ramp slope.

   If your answer is positive, design such a controller and draw the block diagram of the closed-loop system (you have to verify that the requirements would be satisfied). If your answer is negative, explain the reason.

2. Consider the closed-loop system shown in Figure 1 below, where \( K \) is a real constant gain (to be determined) and \( G(s) \) is a real transfer function (i.e., it satisfies \( G(-s) = G(s) \), where \( \overline{s} \) indicates complex conjugate). The magnitude (in dB) and the phase (in degrees) of \( G(-1 + j\omega) \) vs. \( \omega \) are plotted in Figure 2 (see over leaf). Also consider the region \( \mathcal{R} := \{ s \mid \text{Re}(s) < -1 \} \), where \( \text{Re}(s) \) indicates the real part of \( s \). It is known that \( G(s) \) has exactly one pole outside the region \( \mathcal{R} \).

   Except at that pole, \( G(s) \) is bounded and analytic outside the region \( \mathcal{R} \). It is desired to move all the closed-loop poles (of the system shown in Figure 1) inside the region \( \mathcal{R} \).

   a) (5 points) Describe why one may want to move all the closed-loop poles inside the region \( \mathcal{R} \).
   b) (5 points) Determine a Nyquist contour to test for the number of closed-loop poles which lie outside the region \( \mathcal{R} \). Sketch this contour.
   c) (10 points) Draw the Nyquist graph (i.e., the locus of \( KG(s) \) as \( s \) is varied over the Nyquist contour in the clockwise direction) corresponding to the above determined Nyquist contour for \( K = 1 \).
   d) (10 points) From the Nyquist graph obtained above, determine the number of closed-loop poles outside the region \( \mathcal{R} \) for \( K = 1 \).
   e) (10 points) From the Nyquist graph obtained above, determine whether it is possible to move all the closed-loop poles inside the region \( \mathcal{R} \) by choosing \( K \). If so, determine all values of \( K \) such that there are no closed-loop poles outside the region \( \mathcal{R} \). If not, explain why.

   ![Figure 1: Closed-loop system for Question 2.](image-url)
Figure 2: Magnitude and phase plots of $G(-1 + j\omega)$ vs. $\omega$ for Question 2.
1. For the output, $y$, to track the reference, $r$, the closed-loop system must be in a unity-feedback configuration:

$$
\begin{array}{c}
\text{r} \\
\downarrow \\
\text{e} \\
\Downarrow \\
\text{C(s)} \\
\downarrow \\
\text{u} \\
\Downarrow \\
\text{G(s)} \\
\rightarrow \\
\text{y}
\end{array}
$$

where $C(s)$ denotes the controller transfer function (as stated in the question, it is assumed that the sensor (which measures $y$) and the actuator (which applies $u$) have unity transfer functions, thus we do not show them explicitly in the above block diagram). To satisfy requirement (i) all closed-loop poles must have negative real parts with no unstable pole-zero cancellations. Once the closed-loop system is stable (i.e., requirement (i) is satisfied), to satisfy requirement (ii) the open-loop transfer function must have a pole at the origin (i.e., the system should be type 1) [to track constant references] and a couple of poles at $\pm j\pi$ [to reject sinusoidal disturbances with angular frequency $\omega = 2\pi f = \pi$ rad/sec, where $f = 0.5$ Hz, as stated in the question]. To satisfy requirement (iii) the closed-loop poles must have real parts not greater than $-1$ (i.e., the closed-loop poles should be on or to the left of the vertical line passing through $-1$ on the complex plane). Finally, to satisfy requirement (iv) the magnitude of the open-loop dc-gain, i.e., the velocity error coefficient $K_v := \lim_{s \to 0} sG_{OL}(s)$, where $G_{OL}(s) = G(s)C(s)$, should be greater than or equal to 10 in absolute value, since the steady-state error to a unit ramp reference is equal to $\frac{1}{K_v}$ for a type 1 system (alternatively, you can design a type 2 or higher system, i.e., include two or more open-loop poles at the origin, in which case the steady-state error would be zero).

a) In this case the plant has a zero at the origin. Therefore, we can not include an open-loop pole at the origin without unstable pole-zero cancellations. Hence, it is not possible to satisfy requirements (i) and (ii) simultaneously.

b) In this case the plant does not have any zeros at the origin and a controller can be designed to solve the problem. Let us try $C(s) = \frac{K(s+2)^3}{2s(s^2 + \pi^2)}$. Here, the pole at zero is included to achieve the tracking requirement, the poles at $\pm j\pi$ are included to achieve the disturbance rejection requirement, the factor 2 in the denominator is included to cancel the factor 2 in the numerator of $G(s)$ [if you do not include this term, then you have to draw the root locus graph for $k = 2K$ rather than for $K$], and the triple zero at $-2$ is included to attract the closed-loop poles to the left of the vertical line through $-1$ [you can of course choose any three zeros to the left of the vertical line through $-1$ to achieve this]. With this controller, the open-loop system will have one pole at the origin and a couple of poles at $\pm j\pi$, and hence requirement (ii) will be satisfied once the closed-loop system is made stable. The loci of the closed-loop poles as $K$ is varied from 0 to $\infty$ is shown in Figure 3 (see the last page). It is seen that, for a sufficiently large $K$, all the closed-loop poles may be driven to the left of the vertical line passing through $-1$ on the complex plane; and hence, requirements (i) and (iii) can be satisfied. To determine the necessary gain for this [i.e., $K_3 := \max(K_1,K_2)$, where $K_1$ and $K_2$ are as indicated in Figure 3], we can use the Routh-Hurwitz test. For this, first note that the closed-loop characteristic polynomial is $d(s) = s(s^2 + \pi^2)(s-2)+K(s+2)^3 = s^4+(K-2)s^3+(6K+\pi^2)s^2+(12K-2\pi^2)s+8K$, which must
have all the roots on or to the left of the vertical line passing through –1; equivalently the polynomial \( p(s) = d(s-1) = s^4+(K-6)s^3+(3K+12+\pi^2)s^2+(3K-10-4\pi^2)s+(K+3+3\pi^2) \) must have all the roots on or to the left of the region of the imaginary axis. By applying the Routh-Hurwitz test to \( p(s) \), we find that, to satisfy requirement (iii) [and also (i)], we must have 

\[ K \geq K_3 = 19.422. \]

Finally, to satisfy requirement (iv), we must have \( |K_v| \geq 10 \), where 

\[ K_v := \lim_{s \to 0} sG_{OL}(s) = -\frac{4K}{\pi^2}. \]

Hence we must have \( |K| \geq K_4 = \frac{10\pi^2}{4} = 24.674. \) Therefore, to satisfy all the requirements, we have to choose 

\[ K \geq \max(K_3, K_4) = 24.674. \]

For example, we can choose \( K = 25 \), in which case the designed controller, 

\[ C(s) = \frac{25(s+2)^3}{2s(s^2 + \pi^2)}, \]

satisfies all the requirements. The block diagram of the closed-loop system is as shown above.

**Note:** Rather than finding all \( K \) which satisfy requirement (iii), you can first find a \( K \) which satisfies requirement (iv) and test whether this \( K \) satisfies requirement (iii) - if not, increase \( K \) until it satisfies requirement (iii).

2. a) By having all the closed-loop poles inside the region \( \mathcal{R} \), all the closed-loop poles will have real parts less than –1. Therefore, the closed-loop system will be stable. There will also be some stability robustness, since the poles will not be too close to the imaginary axis. Furthermore, by having the real part of all the closed-loop poles less than –1, the response of the closed-loop system will converge to its steady-state at a rate faster than that of \( e^{-t} \).

b) For this, the Nyquist contour must enclose the region of the complex plane which is outside the region \( \mathcal{R} \). Thus, the Nyquist contour must consist of the vertical line passing through –1 and a semi circle of infinite radius enclosing the half-plane to the right of the vertical line passing through –1. This contour is shown in Figure 4 on the last page.

c) The Nyquist graph is shown in Figure 5 on the last page. For \( \omega = 0 \) it starts at \(-a\), where 

\[ a = |G(-1)|. \]

From Figure 2, \( |G(-1)| = 32 \) dB. Thus, \( a = 10^{32/20} \approx 40 \). The graph starts at \(-a\), rather than \(a\), since the phase of \( G(-1) \) is –180 degrees rather than 0 degrees. Since the phase of \( G(-1 + j\omega) \) is greater than –180 degrees for up to \( \omega = \omega_c \approx 9 \) rad/sec, the graph draws a curve below the real line from \(-a\) to \(-b\), where \( b = |G(-1 + j\omega_c)| \). From Figure 2, \( |G(-1 + j\omega_c)| \) is about 8 dB. Thus, \( b = 10^{8/20} \approx 2.5 \). Since for \( \omega > \omega_c \), the phase of \( G(-1 + j\omega) \) is less than –180 degrees and converges to –270 degrees as \( \omega \to \infty \), while the gain drops to zero (i.e., \(-\infty \) dB), the graph draws a curve above the real line from \(-b\) to 0. This curve approaches to zero along the positive imaginary axis since the phase goes to –270 degrees (which is equivalent to 90 degrees) as \( \omega \to \infty \). This completes the graph as the Nyquist contour is traversed from \(-1\) to \(-1 + j\infty \). As the semi circle of infinite radius is traversed, the graph stays at zero, since, from Figure 2, \( \lim_{s \to \infty} G(s) = 0 \). As the rest of the Nyquist contour, i.e., from point \(-1 - j\infty\) to \(-1\) is traversed, the symmetric image about the real axis of the previous graph is obtained since \( G(s) \) is real.

d) The Nyquist graph, which is shown in Figure 5, encircles the \(-1\) point \( N = 1 \) time in the clockwise direction. Thus the number of closed-loop poles which are outside the region \( \mathcal{R} \) is equal to \( N + P = 2 \), where \( P = 1 \) is the number of open-loop poles (i.e., the poles of \( G(s) \)) which are outside the region \( \mathcal{R} \).

e) To have no closed-loop poles outside the region \( \mathcal{R} \), we must have \( N + P = 0 \) or \( N = -1 \). That is the Nyquist graph must encircle the \(-1\) point \( N = -1 \) times in the clockwise direction, i.e., one time in the counter-clockwise direction. This can be achieved by decreasing the open-loop gain by more that \( b \) but by less than \( a \). That is we can move all the closed-loop poles inside the region \( \mathcal{R} \) by choosing 

\[ b < \frac{1}{K} < a \text{ or } \frac{1}{a} = 0.025 < K < 0.4 = \frac{1}{b}. \]
Figure 3: Root-locus for Question 1.

Figure 4: Nyquist contour for Question 2.

Figure 5: Nyquist graph for Question 2.