0. (1 point) Write the group you are registered to on the top-right corner of your answer sheet.

1. (4 points) Consider the system with the transfer function \( G(s) = \frac{s + 1}{s^2 - 2s + 2} \). Is the system stable (in BIBO sense)? Why?

2. (15 points) Consider the closed-loop system shown in Figure 1, where \( G(s) \) is as given in Question 1 and \( C(s) = K \) (a constant gain). Draw the loci of the poles of the closed-loop system on the complex plane as (i) \( K \) is varied from 0 to \( \infty \); (ii) \( K \) is varied from 0 to \( -\infty \). You should indicate the asymptotes (if any) as \( K \rightarrow \pm \infty \); you should calculate the break-away and break-in points (if any), the angles of departure and arrival from/to complex “open-loop poles” and “open-loop zeros” (if any), and the points where the root loci crosses the imaginary axis.

3. (15 points) Consider the closed-loop system described in Question 2. Determine all values of \( K \) which make the closed-loop system stable.

4. (15 points) Find the largest value of \( \sigma \) such that all the closed-loop poles of the system described in Question 2 can be collected in \( C^-_\sigma := \{ s \mid \text{Re}(s) \leq -\sigma \} \) by a choice of \( K \). Also find the value of \( K \) to achieve this.

5. (15 points) Consider the closed-loop system described in Question 2. Let \( K \) be as chosen in Question 4 (if you can not solve Question 4, let \( K \) be any stabilizing gain and indicate the \( K \) you choose). Sketch the unit step response (the output \( y \) when a unit step is applied to the reference \( r \)) of the closed-loop system (do not calculate actual response; sketch your best guess from pole/zero locations). Calculate the steady-state output and the steady-state error and indicate these on your sketch. Is it possible to choose \( K \) such that the output \( y \) tracks the reference \( r \) with no steady-state error? Why?

6. (20 points) Design a rational and proper controller \( C(s) \) for the system shown in Figure 1 (where \( G(s) \) is as given in Question 1), such that there will be no steady-state error in response to a step reference.

7. (10 points) Determine the steady-state error of the closed-loop system shown in Figure 1, where \( G(s) \) is as given in Question 1 and \( C(s) \) is as designed in Question 6, in response to (i) a unit ramp reference, \( r(t) = t, t \geq 0 \); (ii) a unit parabolic reference, \( r(t) = \frac{1}{2}t^2, t \geq 0 \).

8. (5 points) Let \( G(s) = \frac{s}{s^2 - 2s + 2} \). Is it possible to design a rational and proper controller \( C(s) \) for the system shown in Figure 1 such that there will be no steady-state error in response to a step reference and that the closed-loop system is internally stable? Why?
1. Since there are both positive and negative coefficients in the denominator polynomial, there is at least one pole of \( G(s) \) in the closed right half complex plane (CRHP). Therefore, the system is not stable. The actual location of the poles (i.e., the roots of the denominator polynomial which are not cancelled by any roots of the numerator polynomial) can also be found as \( 1 \pm j \).

2. The closed-loop poles are the roots of \( p(s) = a(s) + K b(s) \), where \( a(s) = s^2 - 2s + 2 \) and \( b(s) = s + 1 \).

The open-loop poles (roots of \( a(s) \)) are at \( 1 \pm j \) and the open-loop zero (root of \( b(s) \)) is at \(-1\). Since the relative degree (number of open-loop poles minus the number of open-loop zeros) is one, there is only one asymptote, which is along the negative real axis for \( K > 0 \), and along the positive real axis for \( K < 0 \). There are two branches, each starting from one of the open-loop poles, then they break-in to the real axis, and then one tends to infinity along the relevant asymptote (along the negative real axis for \( K > 0 \), and along the positive real axis for \( K < 0 \)) and the other tends to the open-loop zero at \(-1\). The part of the real axis to the left of \(-1\) is a part of root-loci for \( K > 0 \) and the part to the right is a part of root-loci for \( K < 0 \) (thus the break-in point must be to the left of \(-1\) for \( K > 0 \) and must be to the right of \(-1\) for \( K < 0 \)). The break-in points are the roots of \( a'(s)b(s) - b'(s)a(s) = s^2 + 2s - 4 = 0 \), which are \( s_{a,b} = -1 \pm \sqrt{5} \). Thus, \( s_a = -1 - \sqrt{5} \approx -3.236 \) is the break-in point for \( K > 0 \) and \( s_b = -1 + \sqrt{5} \approx 1.236 \) is the break-in point for \( K < 0 \). The angle of departure from the open-loop pole at \( 1+j \) for \( K > 0 \) is \( \phi_{d1}^+ = \tan^{-1}(\frac{1}{2}) - 90^\circ + (2k+1)180^\circ = 116.565^\circ \) (for \( k = 0 \)). Since the root-loci is symmetric about the real axis, the angle of departure from the open-loop pole at \( 1-j \) for \( K > 0 \) is \( \phi_{d2}^+ = -\phi_{d1}^+ = -116.565^\circ \). Furthermore, since the root-loci must be smooth, the angle of departures from these open-loop poles for \( K < 0 \) must be \( \phi_{d1}^- = \pm 180^\circ + \phi_{d1}^+ = -63.435^\circ \) and \( \phi_{d2}^- = \pm 180^\circ + \phi_{d2}^+ = 63.435^\circ \). To find the values of \( K \) at which the root-loci cross the imaginary axis and the corresponding crossing points, apply the Routh-Hurwitz test to \( p(s) = a(s) + K b(s) = s^2 + (K-2)s + (K+2) \). This gives that there are a couple of complex conjugate poles on the imaginary axis for \( K = 2 \) and a pole at the origin for \( K = -2 \). For \( K = 2 \), \( p(s) = s^2 + 4 \), whose roots are \( \pm 2j \). Thus the root-loci crosses the imaginary axis at \( \pm 2j \) for \( K > 0 \) (with \( K = 2 \)) and at the origin for \( K < 0 \) (with \( K = -2 \)). The root-locus graphs for \( K > 0 \) and \( K < 0 \) are respectively shown in Figures 2a and 2b.

Fig. 2a. The root-locus graph for \( K > 0 \).

Fig. 2b. The root-locus graph for \( K < 0 \).
3. From Fig. 2a, both branches are in the open left half complex plane (OLHP) for $K > 2$ and in the CRHP for $0 \leq K \leq 2$. From Fig. 2b, there is always at least one branch in the CRHP for $K < 0$. Therefore, the closed-loop system is stable for $K > 2$ and unstable for $K \leq 2$.

4. From Fig. 2a, $\sigma = -s_a = 1 + \sqrt{5}$. The corresponding $K$ can now be found by using the distance formula:

$$K = \frac{(\sqrt{(2 + \sqrt{5})^2 + 1^2})^2}{\sqrt{5}} = \frac{10 + 4\sqrt{5}}{\sqrt{5}} = 4 + 2\sqrt{5} \approx 8.472$$

5. The closed-loop transfer function (from $r$ to $y$) is

$$G_c(s) = \frac{KG(s)}{1 + KG(s)} = \frac{K(s + 1)}{s^2 + (K - 2)s + (K + 2)} = \frac{K(s + 1)}{(s + \sigma)^2}$$

where $K = 4 + 2\sqrt{5} \approx 8.472$ and $\sigma = 1 + \sqrt{5} \approx 3.236$. Therefore, the closed-loop system has one zero at $-1$ and a double pole at $-\sigma \approx -3.236$. The double pole at $-\sigma$ causes a critically damped response with a time constant $\frac{1}{\sigma} \approx 0.309\text{tu}$ (where tu stands for time unit). The zero at $-1$, does not change the steady-state response, but causes a strong overshoot (since it is in the OLHP and closer to the imaginary axis than the poles). The steady-state value of the output is:

$$y_{ss} = \lim_{t \to \infty} y(t) = \lim_{s \to 0} sG_c(s) = \lim_{s \to 0} sG_c(0) = \frac{K}{\sigma^2} = \frac{4 + 2\sqrt{5}}{(1 + \sqrt{5})^2} = \frac{1}{4} + \frac{1}{4\sqrt{5}} \approx 0.809$$

Thus, the steady-state error is

$$e_{ss} = 1 - y_{ss} = \frac{3}{4} - \frac{1}{4\sqrt{5}} \approx 0.191$$

The unit step response is as follows:

Since the open-loop system is of type zero for any constant $K$, it is not possible to track a constant reference with no steady-state error.
6. In order not to have any steady-state error in response to a step reference, the open-loop system must have at least one pole at the origin and the closed-loop system must be stable. Since $G(s)$ has no poles at the origin, we must include such a pole in $C(s)$ and this $C(s)$ must also stabilize the closed-loop system. Let $C(s) = \frac{K(s+a)}{s}$ (if this does not work, we should add more poles and zeros to $C(s)$). By considering the root-locus graphs, a good choice for $a$ is $a = 2$. For this $a$, the closed-loop transfer function is

$$G_c(s) = \frac{G(s)C(s)}{1 + G(s)C(s)} = \frac{K(s^2 + 3s + 2)}{s^3 + (K - 2)s^2 + (3K + 2)s + 2K}$$

By applying the Routh-Hurwitz test to the denominator polynomial of $G_c(s)$, it is found that the closed-loop system is stable for $K > 2$. If we let $K = 4$, for example, we obtain

$$C(s) = \frac{4s + 8}{s} = 4 + \frac{8}{s}$$

which is a proportional-plus-integral (PI) controller with proportional gain 4 and integral gain 8. This controller can be realized as (not required for the exam):

$$\dot{x}(t) = e(t)$$

$$u(t) = 8x(t) + 4e(t)$$

where $x$ is the state of the controller (the output of the integrator) and $e(t) = r(t) - y(t)$ is the tracking error.

7. (i) With $G(s)$ as in Question 1 and $C(s)$ as designed in Question 6, we have $K_v = \lim_{s \to 0} s G(s) C(s) = 4$. Therefore, in response to a unit ramp reference, the steady-state error is

$$e_{ss} = \lim_{s \to 0} \frac{s}{s^2 (1 + G(s) C(s))} = \frac{1}{K_v} = \frac{1}{4}$$

(ii) With $G(s)$ as in Question 1 and $C(s)$ as designed in Question 6, the system is a type 1 system, which can not follow any polynomial reference of degree greater than 1. The steady-state error in response to a unit parabolic reference is

$$e_{ss} = \lim_{s \to 0} \frac{s}{s^3 (1 + G(s) C(s))} = \infty$$

8. In order not to have any steady-state error in response to a step reference, the open-loop system must have at least one pole at the origin and the closed-loop system must be stable. Since $G(s)$ has one zero at the origin, however, in order for the open-loop transfer function $G_o(s) = G(s)C(s)$ to have a pole at the origin, one must include at least two poles in $C(s)$. This, however, would result in an unstable pole-zero cancellation which will make the closed-loop system internally unstable. Therefore, due to the zero of $G(s)$ at the origin, it is not possible to design a controller which will result in no steady-state error and will make the closed-loop system internally stable.