0. (2 points) Write the group you are registered to on the top-right corner of your answer sheet.

1. Consider the scalar system described by

\[
\dot{x}(t) = x(t) + u(t) + d(t) \\
y(t) = x(t) + u(t)
\]

where \(x\), \(u\), \(d\), and \(y\) are, respectively, the state, the control input, the disturbance, and the output. Is it possible to design a linear time-invariant (LTI) controller for this system so that the output will track constant references with no steady-state error and the steady-state error in response to a ramp reference will not exceed (in absolute value) 10% of the ramp slope; despite sinusoidal disturbances of angular frequency 1 rad/sec (where sec is the unit of time) with a DC (i.e., constant) off-set? If not, explain why. If so, design such a controller; write the controller transfer function and draw a block diagram of the overall closed-loop system.

2. Repeat Question 1 for the system

\[
\dot{x}(t) = x(t) + u(t) + d(t) \\
y(t) = 3x(t) + u(t)
\]

3. (25 points) Consider the following system:

\[
\begin{array}{c}
\text{u} \\
\downarrow \\
\text{+} \\
\downarrow \\
\text{K}_1 \int \\
\downarrow \\
\text{+} \\
\downarrow \\
\text{K}_3 \\
\downarrow \\
\text{+} \\
\downarrow \\
\text{K}_2 \int \\
\downarrow \\
y
\end{array}
\]

where “\(\int\)” indicates an integrator. Choose the gains \(K_1\), \(K_2\), and \(K_3\) such that (i) the system is stable, (ii) steady-state value of the output is 2 in response to a unit step input, (iii) maximum percent overshoot is 10%, and (iv) the output settles to 2% of its steady-state value within 10 seconds. Write the transfer function for the chosen \(K_1\), \(K_2\), and \(K_3\) and sketch the unit step response.
1. The transfer function of the system from the control input \( u \) to the output \( y \) is 
\[
G(s) = \frac{s}{s - 1}
\]
Since the system has a zero at zero, it is not possible to design a controller which will achieve 
tracking of constant references and internal stability (note that, without internal stability, it will 
not be possible to achieve tracking due to disturbances and unknown initial conditions).

2. The transfer function of the system from the control input \( u \) to the output \( y \) is 
\[
G(s) = \frac{s + 2}{s - 1}
\]
Since the system does not have any zeros at zero and/or at \( \pm j \), it is possible to design a controller 
which will achieve tracking of constant references and rejection of sinusoidal disturbances of 
angular frequency 1 rad/sec with a DC off-set. For this, the controller must have poles at zero 
and at \( \pm j \), must stabilize the closed-loop system, and must be implemented in a unity feedback 
configuration as shown in Figure 1. With one open-loop pole at zero, the steady-state error in 
response to a ramp reference will not exceed (in absolute value) 10\% of the ramp slope provided 
that the closed-loop system is stable and the open-loop gain of the system is at least \( \frac{1}{10} = 10 \).

Let us try a controller with transfer function 
\[
C(s) = \frac{n(s)}{s(s^2 + 1)},
\]
where \( n(s) \) is a polynomial of degree at most 3 (so that the controller is proper). If this is not 
sufficient, we can add more poles and increase the degree of \( n(s) \). If we let \( n(s) = K \) (i.e., a 
constant), the loci of the closed-loop poles, for \( K > 0 \), will be as shown in Figure 2a. Clearly, it 
is not possible to stabilize the overall system with \( K > 0 \). A similar analysis will show that the 
same conclusion also holds for \( K < 0 \). Let us choose \( n(s) = K(s + 1)^2 \). Then, the loci of the 
closed-loop poles, for \( K > 0 \), will be as shown in Figure 2b. In this case, it is possible to move all 
the closed-loop poles into the left-half plane with a sufficiently large \( K \). By applying the Routh 
test, the closed-loop system is stable for \( K > \frac{10 + \sqrt{73}}{9} \approx 2.06 \). To satisfy the requirement 
that the steady-state error in response to a ramp reference will not exceed (in absolute value) 10\% of 
the ramp slope, we must have \( |K_v| \geq 10 \), where \( K_v = \lim_{s \to 0} sG(s)C(s) = -2K \). Thus we must 
have \( K \geq 5 \), which also satisfies the stability requirement. By choosing \( K = 5 \), the controller 
transfer function is 
\[
C(s) = \frac{5(s + 1)^2}{s(s^2 + 1)}
\]
The block diagram of the overall closed-loop system is shown in Figure 1.

![Figure 1: Overall closed-loop system.](image1)

![Figure 2: Root loci for (a) \( n(s) = K \); (b) \( n(s) = K(s + 1)^2 \).](image2)
The designed controller can be realized as (not required for the exam):

\[
\dot{z}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} z(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e(t)
\]

\[
u(t) = \begin{bmatrix} 5 & 10 & 5 \end{bmatrix} z(t)
\]

where \(e(t) = r(t) - y(t)\), where \(r\) is the reference, and \(z\) is the state of the controller.

3. By replacing each integrator by its transfer function, \(\frac{1}{s}\), and using block diagram reduction techniques, the transfer function from \(u\) to \(y\) is obtained as \(\frac{K_1 K_2}{s^2 + K_1 s + K_1 K_2 K_3}\). Therefore, for stability, we must have \(K_1 > 0\) and \(K_1 K_2 K_3 > 0\). By equating the obtained transfer function to the standard second order transfer function, \(\frac{K \omega_n^2}{s^2 + 2 \zeta \omega_n s + \omega_n^2}\), we determine that the DC gain is \(K = \frac{1}{K_3}\). Therefore, to have a steady-state output value of 2 in response to a unit step input, we must have \(\frac{1}{K_3} = 2\) or \(K_3 = \frac{1}{2}\). To have a maximum percent overshoot of 10%, we must have \(\zeta = \frac{\left|\ln(0.1)\right|}{\sqrt{\pi^2 + \left|\ln(0.1)\right|^2}} = 0.59\). For the output to settle to 2% of its steady-state value within 10 seconds, we must have (we use the envelope settling time) \(\zeta \omega_n \geq \frac{1}{10} \ln \left(0.02 \sqrt{1 - \zeta^2}\right) = 0.4127\). Since \(K_1 = 2 \zeta \omega_n\), we must have \(K_1 \geq 0.8254\). Let \(K_1 = 1\). Then, to determine \(K_2\), note that \(K_1 = 2 \zeta \omega_n = 2 \zeta \sqrt{K_1 K_2 K_3}\). Hence, \(K_2 = \frac{K_1}{4 \zeta^2 K_3} = 1.4308\). The overall transfer function is \(\frac{1.4308 s^2 + s + 0.7154}{s^2 + s + 0.7154}\). The unit step response is as shown in Figure 3. To draw a sufficiently accurate graph, you need to note that (i) the steady-state value is 2; (ii) the peak time is \(t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n \sqrt{1 - \zeta^2}} = 4.605\) seconds, at which time the peak value is 2.2 (10% over the steady-state value); (iii) the period of damped oscillations is \(T = 2t_p = 9.21\) seconds; (iv) the output settles to \(\pm 2\%\) of its steady state value (i.e., between 1.96 and 2.04) in less than 10 seconds (the actual 2% settling time is about 7 seconds).

![Figure 3: Step response.](image-url)