0. (1 point) Write the group you are registered to on the top-right corner of your answer sheet.

1. (4 points) Consider the system with the transfer function \( G(s) = \frac{2s + 1}{(s^2 + 2s + 2)(s - 1)} \). Is the system stable (in BIBO sense)? Why?

2. (20 points) Consider the closed-loop system shown in Figure 1, where \( G(s) \) is as given in Question 1 and \( C(s) = K \) (a constant gain). Draw the loci of the poles of the closed-loop system on the complex plane as (i) \( K \) is varied from 0 to \( \infty \); (ii) \( K \) is varied from 0 to \( -\infty \). You should indicate the asymptotes (if any) as \( K \to \pm\infty \); you should calculate the break-away and break-in points (if any), the angles of departure and arrival from/to complex “open-loop poles” and “open-loop zeros” (if any), and the points where the root loci crosses the imaginary axis.

3. (15 points) Consider the closed-loop system described in Question 2. Determine all values of \( K \) which make the closed-loop system stable.

4. (15 points) Find the largest value of \( \sigma \) such that all the closed-loop poles of the system described in Question 2 can be collected in \( C^-_{\sigma} := \{s \mid \text{Re}(s) \leq -\sigma\} \) by a choice of \( K \). Also find the value of \( K \) to achieve this.

5. (15 points) Consider the closed-loop system described in Question 2. Let \( K \) be as chosen in Question 4 (if you can not solve Question 4, let \( K \) be any stabilizing gain and indicate the \( K \) you choose). Sketch the unit step response (the output \( y \) when a unit step is applied to the reference \( r \)) of the closed-loop system (do not calculate actual response; sketch your best guess from pole/zero locations) Calculate the amount of the steady-state error and indicate this on your sketch. Is it possible to choose \( K \) such that the output \( y \) tracks the reference \( r \) with no steady-state error? Why?

6. (20 points) Design a rational and proper controller \( C(s) \) for the system shown in Figure 1 (where \( G(s) \) is as given in Question 1), such that there will be no steady-state error in response to a step input.

7. (10 points) Determine the steady-state error of the closed-loop system shown in Figure 1, where \( G(s) \) is as given in Question 1 and \( C(s) \) is as designed in Question 6, in response to (i) a unit ramp reference, \( r(t) = t, t \geq 0 \); (ii) a unit parabolic reference, \( r(t) = \frac{1}{2}t^2, t \geq 0 \).

![Figure 1: Closed-loop system](image-url)
SOLUTIONS

1. The transfer function has a pole in the closed right half complex plane (CRHP) (at $s = 1$). Therefore, the system is not stable.

2. The closed-loop poles are the roots of $p(s) = a(s) + kb(s)$, where $a(s) = (s^2 + 2s + 2)(s - 1) = s^3 + s^2 - 2$, $b(s) = s + \frac{1}{K}$, and $k = 2K$. The open-loop poles (roots of $a(s)$) are at $+1$ and $-1 \pm j\sqrt{2}$ and the open-loop zero (root of $b(s)$) is at $-\frac{1}{2}$.

   (i) For $K > 0$, $k > 0$, the center of the asymptotes is at $\sigma_A = \frac{1+(-1)+(-1)-(-\frac{1}{2})}{3-1} = -\frac{1}{4}$. The angles of the asymptotes are $\theta_A = \frac{(2l+1)180^\circ}{3-1} = \pm 90^\circ$. The branch which starts at $+1$ tends to the open-loop zero at $-\frac{1}{2}$, as $k \to +\infty$ and the branches which start at $-1 \pm j$ tend to infinity along these asymptotes. There are no break-away/break-in points for $k > 0$. The angle of departure from the open-loop pole at $-1 + j$ is $\phi_{d1}^+ = (180^\circ - \tan^{-1}(2)) - (180^\circ - \tan^{-1}(\frac{1}{2})) = 53^\circ$. The only point where the root loci crosses the imaginary axis is at the origin. At this point $k = \frac{1+\sqrt{2}+\sqrt{2}}{2} = 4$. Thus, $K = \frac{5}{\frac{3}{2}} = 2$. The root-locus graph is shown in Fig. 2a.

   (ii) For $K < 0$, $k < 0$, the angles of the asymptotes are $\theta_A = \frac{(2l)180^\circ}{3-1} = 0^\circ$ and $180^\circ$. The branch which starts at $+1$ tends to infinity along the positive real axis (the asymptote with $\theta_A = 0^\circ$) as $k \to -\infty$. The branches which start at $-1 \pm j$ break-in the real axis at the real root of $a'(s)b(s) - b'(s)a(s) = 2s^3 + \frac{5}{2}s^2 + s + 2 = 0$ which is $s_1 = -1.4$. Then, one of the branches tends to the open-loop zero at $-\frac{1}{2}$ and the other tends to infinity along the negative real axis (the asymptote with $\theta_A = 180^\circ$) as $k \to -\infty$. The angle of departure from the open-loop pole at $-1 + j$ is $\phi_{d1}^- = 180^\circ + \phi_{d1} = 233^\circ$. Similarly, the angle of departure from the open-loop pole at $-1 - j$ is $\phi_{d2}^- = -\phi_{d1} = -180^\circ + \phi_{d2}^+ = -233^\circ$. The root loci does not cross the imaginary axis for $k < 0$. The root-locus graph is shown in Fig. 2b.

3. From Fig. 2a, all the branches are in the open left half complex plane (OLHP) for $K > 2$ and there is one branch in the CRHP for $K \leq 2$. From Fig. 2b, there is always one branch in the CRHP. Therefore, the closed-loop system is stable for $K > 2$ and unstable for $K \leq 2$. 

![Fig. 2a. The root-locus graph for $K > 0$.](image)

![Fig. 2b. The root-locus graph for $K < 0$.](image)
4. Let \( q(s) = p(s - \sigma) = s^3 + (1 - 3\sigma)s^2 + (2K - 2\sigma + 3\sigma^2)s + K - 2 - 2K\sigma + \sigma^2 - \sigma^3 \). As seen from Fig. 2a, one of the right-most roots of \( p(s) \) must be real when the real part of its right-most roots are at \(-\sigma\). Thus, \( q(s) \) must have a root at the origin, i.e., the constant term of \( q(s) \) must be zero. Therefore,

\[
K = \frac{2 - \sigma^2 + \sigma^3}{1 - 2\sigma}
\]  

(1)

Furthermore, from Fig. 2a, \( p(s) \) must also have a couple of complex conjugate roots with real part at \(-\sigma\), when the real part of its right-most roots are at \(-\sigma\). Thus, \( q(s) \) must have a couple of complex conjugate roots on the imaginary axis. Therefore, in the Routh table of \( q(s) \), there should be a zero term in the first column such that the terms above and below it have the same sign. Since the constant term of \( q(s) \) is zero, \( q(s) \) can be reduced to a second degree polynomial by dividing by \( s \). Therefore, the first column elements of \( q(s) \) are its coefficients. This means that the second coefficient of \( q(s) \), which is \( 1 - 3\sigma \), must be zero and the first and the third coefficients must have the same sign. Thus, \( \sigma = \frac{1}{3} \). The corresponding \( K \) can now be found from (1) as \( K = \frac{52}{9} \). Note that for this \( \sigma \) and \( K \), the first and the third coefficients of \( q(s) \) are both positive.

5. As explained above, with \( K = \frac{52}{9} \), one closed-loop pole is at \(-\sigma = -\frac{1}{3}\) and the remaining two are at \(-\sigma \pm j\omega_d\). To find \( \omega_d \), note that for \( K = \frac{52}{9} \) and \( \sigma = \frac{1}{3} \), \( q(j\omega) = -j\omega(\omega^2 - \frac{101}{3}) \), the roots of which (apart from the zero root) are \( \pm\frac{\sqrt{101}}{\sqrt{3}} \). Thus, \( \omega_d = \frac{\sqrt{101}}{\sqrt{3}} = 3.35 \). Therefore, the most dominant part in the transient response is the response due to the couple of complex conjugate poles at \(-\frac{1}{3} \pm 3.35j\), i.e., an underdamped response of a second order system with \( \omega_n = \frac{\sqrt{102}}{3} \) and \( \zeta = \frac{1}{\sqrt{102}} = 0.1 \). Thus, the response will include underdamped oscillations at frequency \( \omega_d = 3.35 \text{ rad/tu} \) (tu stands for time unit). These oscillation will be damped more than that of a second order system with \( \zeta = 0.1 \), because of the real pole at \(-\frac{1}{3}\). The effect of this pole, however, will be lessened by the zero at \(-\frac{1}{2}\) (the effect of the pole will be more than that of the zero, since it is closer to the imaginary axis). The period of the damped oscillations will be \( \frac{2\pi}{\omega_d} = 1.876 \text{ tu} \). There will be a peak at time \( \frac{\pi}{\omega_d} = 0.938 \text{ tu} \). The overshoot at the first peak will be less than \( M_p = e^{-\zeta\pi/\sqrt{1-\zeta^2}} = 73\% \), due to the real pole at \(-\frac{1}{3}\). To find the steady-state error, note that \( K_p := \lim_{s \to 0} KG(s) = \frac{K}{s^2} = -\frac{52}{9} = -2.89 \). Thus, the steady-state error in response to a unit step reference is \( \epsilon_{ss} = \lim_{s \to 0} \frac{s}{s(1+KG(s))} = \frac{1}{1+K_p} = -0.53 \). Therefore, the steady-state value of the output is \( y_{ss} = 1 - \epsilon_{ss} = 1.53 \). The unit step response is as follows:
Since the open-loop system is of type zero for any constant $K$, it is not possible to track a constant reference with no steady-state error.

6. In order not to have any steady-state error in response to a step reference, the open-loop system must have at least one pole at the origin and the closed-loop system must be stable. Since $G(s)$ has no poles at the origin, we must include such a pole in $C(s)$ and this $C(s)$ must also stabilize the closed-loop system. Let $C(s) = \frac{K(s+a)}{s}$ (if this does not work, we should add more poles and zeros to $C(s)$). By considering the root-locus graphs, a good choice for $a$ is $a = \frac{1}{2}$. For this $a$, there are two asymptotes with angles $\pm 90^\circ$ and center at the origin. Thus, once all the branches get into the OLHP, they stay there (for larger $a$ the branches may not get into the OLHP at all or even if they get into there, they will move back to the CRHP for larger $K$; for negative $a$ there will be at least one branch in the CRHP; for $0 < a < \frac{1}{2}$, the closed-loop system can be stabilized but putting a zero too close to the origin will lessen the effect of the pole at the origin, making it harder to track any reference). The root-locus graph for $a = \frac{1}{2}$ and $K > 0$ is shown in Fig. 3a (the closed-loop system can not be stabilized for $K < 0$). The values $K$ which stabilize the closed-loop system can be found by the Routh-Hurwitz test as $K > \frac{8}{7}$. By choosing $K = 2$, for example, we obtain $C(s) = \frac{2s+1}{s}$. The unit step response of the closed-loop system for this $C(s)$ is shown in Fig. 3b.

7. (i) With $G(s)$ as in Question 1 and $C(s)$ as designed in Question 6, we have $K_v = \lim_{s \to 0} sG(s)C(s) = \frac{-1}{2}$. Therefore, in response to a unit ramp reference, the steady-state error is $e_{ss} = \lim_{s \to 0} s^2(1+G(s)C(s)) = \frac{1}{K_v} = -2$. (ii) With $G(s)$ as in Question 1 and $C(s)$ as designed in Question 6, the system is a type 1 system, which can not follow any polynomial reference of degree greater than 1. The steady-state error in response to a unit parabolic reference is $e_{ss} = \lim_{s \to 0} s^3(1+G(s)C(s)) = \infty$. 

Fig. 3a. The root-locus graph.

Fig. 3b. The unit step response.