1. (50 points) Consider linear time-invariant (LTI) plants with transfer function
   
   a) \( G(s) = \frac{s}{s-2} \)  
   b) \( G(s) = \frac{s+2}{s-2} \)

   For each plant, determine whether it is possible to design a LTI strictly-proper controller such that:
   i) the closed loop system is internally stable, ii) the output of the plant can track constant reference signals with no steady-state error in the presence of sinusoidal additive disturbances of frequency 0.5 Hz, iii) the output settles to its steady-state value at a rate no slower than that of \( e^{-2t} \), and iv) the steady-state error in response to a ramp reference signal is no more than 5% of the ramp slope. If your answer is positive, design such a controller (you have to verify that the requirements would be satisfied) and draw the block diagram of the closed-loop system. If your answer is negative, explain the reason.

2. (30 points) Consider the system shown below:

   a) Draw the loci of the poles of the closed-loop system on the complex plane as (i) \( K \) is varied from 0 to \( \infty \); (ii) \( K \) is varied from 0 to \( -\infty \). You should indicate the asymptotes (if any) as \( K \rightarrow \pm \infty \); you should calculate the break-away and break-in points (if any), the angles of departure and arrival from/to complex “open-loop poles” and “open-loop zeros” (if any), and the points where the root loci crosses the imaginary axis.

   b) By using the root-locus graph you obtained, find the values of \( K \) for which the closed-loop system is stable.

3. (20 points) Consider the LTI system whose block diagram is shown below. Choose \( K_1, K_2, \) and \( K_3 \) such that the system is stable and, in response to a unit step input, (i) steady-state value of the output is \(-3\); (ii) maximum percent overshoot is 15%; and (iii) the output settles to 10% of its steady-state value within 2 seconds.
1. For the output, \( y \), to track the reference, \( r \), the closed-loop system must be in a unity-feedback configuration:

\[
\begin{align*}
&\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad 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the roots on or to the left of the vertical line passing through \(-2\); equivalently the polynomial 
\[ p(s) = \frac{d(s-2)}{s^3 + (K-10)s^3 + (4K + 36 + \pi^2)s^2 + (4K - 56 - 6\pi^2)s + (32 + 8\pi^2) \] 
must have all the roots on or to the left of the imaginary axis. By applying the Routh-Hurwitz test, we find that, to satisfy this requirement we must have \(K \geq 32.36\). Thus, to satisfy both requirements (iii) and (iv), we must have \(K \geq \text{max}(32.36, 12.34) = 32.36\). Therefore, we can use the above defined controller with any \(K \geq 32.36\). Note that, rather than finding all \(K\) which satisfy requirement (iii), you can try a specific \(K\) (which may be guessed from the root-locus graph) and show that the requirement is satisfied. e.g., for \(K = 35\), 
\[ p(s) = s^4 + 25s^3 + 185.87s^2 + 24.78s + 110.96, \] 
which (by applying the Routh-Hurwitz test) has all the roots in the left-half complex plane. With this \(K\), 
\[ C(s) = \frac{35s^2 + 280s + 560}{s(s^2 + \pi^2)}. \] 
The block diagram of the closed-loop system is as shown above.

2. The closed-loop transfer function (from \(r\) to \(y\)) is 
\[ G_c(s) = \frac{K(s+2)}{(2s^3 + 2s^2 - 4) + K(s+2)}. \]

a) The closed-loop poles are the roots of 
\[ p(s) = (s^3 + s^2 - 2) + \frac{K}{2}(s+2), \] 
where we divided the denominator polynomial by 2 to make it monic. In standart form, we have 
\[ p(s) = a(s) + \kappa b(s), \] 
where \(a(s) = s^3 + s^2 - 2 = (s-1)(s^2 + 2s + 2), b(s) = s + 2, \) and \(\kappa = \frac{K}{2}\). Thus, the open-loop poles (the roots of \(a(s)\)) are at +1 and \(-1 \pm j\), and the open-loop zero (the root of \(b(s)\)) is at \(-2\). The root loci consists of three branches each starting at one of the open-loop poles for \(K = 0\). One of these branches go to the open-loop zero at \(-2\) as \(K\) (equivalently, \(\kappa\)) goes to \(\pm \infty\), the other two go to infinity along the asymptotes with center at \(\sigma_A = \frac{-1 - 1 + 1 - (-2)}{3-1} = \frac{1}{2}\).

i) As \(K\) (equivalently, \(\kappa\)) is varied from 0 to \(\infty\), the angle of the asymptotes are \(\theta_A = \pm \frac{\pi}{2}\) rad = \(\pm 90\) degrees. The branch which starts at the open-loop pole at +1 goes to the open-loop zero at \(-2\) along the real axis. The branch which starts at the open-loop pole at \(-1 + j\) departs from that pole with an angle of 
\[ \phi_d = 180 - \left(180 - \tan^{-1}\left(\frac{1}{2}\right)\right) - 90 + 45 = -18.43 \text{ degrees} \]
and goes to infinity along the asymptote which has the angle of +90 degrees. To find the point where this branch crosses the imaginary axis, we apply the Routh-Hurwitz test to \(p(s)\) and determine that the imaginary axis is crossed when \(\kappa = 2\) at \(j\sqrt{2}\). The branch which starts at the open-loop pole at \(-1 - j\) is symmetric to the branch which starts at the open-loop pole at \(-1 + j\), with respect to the real axis. Thus, it departs with an angle of \(-(-18.43) = 18.43 \text{ degrees}\), crosses the imaginary axis at \(-j\sqrt{2}\) for \(\kappa = 2\) and goes to infinity along the asymptote which has the angle of \(-90\) degrees. The root-locus graph is as shown in Figure 1.

ii) As \(K\) (equivalently, \(\kappa\)) is varied from 0 to \(-\infty\), the angle of the asymptotes are 0 and 180 degrees. The branch which starts at the open-loop pole at +1 goes to infinity along the asymptote which has the angle of 0 degrees (i.e., the positive real axis). The branch which starts at the open-loop pole at \(-1 + j\) departs from that pole with an angle of \(\phi_d = 180 + \phi_d^+ = 161.57 \text{ degrees}\), where \(\phi_d^+\) is the angle of departure for \(\kappa > 0\), and goes to the break-in point at \(\sigma_1 = -2.93\), which is the only real root of 
\[ a'(s)b(s) - b'(s)a(s) = 2s^3 + 7s^2 + 4s + 2 = 0. \] 
At this point, this branch meets with the
branch which comes from the open-loop pole at $-1-j$ (which departs with an angle of $-161.57$ degrees). After that, one of these two branches goes to the open-loop zero at $-2$ along the real axis and the other branch goes to infinity along the negative real axis (i.e., along the asymptote which has the angle of 180 degrees). The root-locus graph is as shown in Figure 2.

Figure 1: The root-locus graph for $\kappa > 0$.

Figure 2: The root-locus graph for $\kappa < 0$. 

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b) For $\kappa < 0$, one of the branches is always in the right-half complex plane. For $\kappa > 0$, the branch which starts at the open-loop pole at $+1$ gets into the left-half plane at the origin. At this point $\kappa = \frac{1 \times \sqrt{2} \times \sqrt{2}}{2} = 1$. The branches which start at the open-loop poles at $-1 \pm j$ leave the left-half complex plane for $\kappa = 2$, as determined in part a. Thus, the closed-loop system is stable for $1 < \kappa < 2$, i.e., for $2 < K < 4$.

3. The transfer function from $u$ to $y$ is $G(s) = \frac{K_1}{s^2 - K_2 s - K_3}$. Therefore, for stability, we need $K_2 < 0$ and $K_3 < 0$. To have a steady-state output of $-3$, in response to a unit step input, we must have $\lim_{s \to 0} sG(s) \frac{1}{s} = \frac{K_1}{-K_3} = -3$, or

$$K_1 = 3K_3.$$  \hfill (1)

Since the system is a second order system with no finite zeros, to have a maximum percent overshoot of 15%, we must have $e^{-\zeta \pi / \sqrt{1 - \zeta^2}} = 0.15$ or $\zeta = \frac{|\ln(0.15)|}{\sqrt{\pi^2 + (\ln(0.15))^2}} = 0.517$, where 

$$2\zeta \omega_n = -K_2 \text{ and } \omega_n^2 = -K_3.$$ Thus,

$$K_2 = -2\zeta \sqrt{-K_3} = -1.034\sqrt{-K_3}.$$ \hfill (2)

Using the envelope settling time, requirement (iii) is satisfied if $\frac{1}{\zeta \omega_n} \ln \left( 0.1 \sqrt{1 - \zeta^2} \right) \leq 2$. With $\zeta = 0.517$, this means $\omega \geq 2.38$ or $K_3 \leq -5.66$. If, for example, we let $K_3 = -6$, from (1) and (2) we find $K_1 = -18$ and $K_2 = -2.53$. 

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