0. (2 points) Write the group you are registered to on the top-right corner of your answer sheet.

1. Consider a metal object of mass \( M \) which can move inside a pipe placed vertically. The position of the object from the bottom of the pipe at time \( t \) is denoted by \( y(t) \). The gravitational acceleration that effects the object is \( g \) (i.e., the gravity pulls down the object by a force \( Mg \)). An electromagnet, which has an input current \( i(t) \), is placed at the top of the pipe and pulls up the object by a force \( ki(t) \), where \( k \) is a constant. While moving inside the pipe, the object is subject to viscous friction of coefficient \( \beta \) (i.e., the friction force exerted on the object is \(-\beta \dot{y}(t))

Assume that the object always stays within the pipe and does not hit the bottom or the top.

a) (5 points) Write the equation of motion of the system, where \( i \) is the input and \( y \) is the output.

b) (8 points) Is this system (i) static or dynamic; (ii) linear or non-linear; (iii) time-invariant or time-varying; (iv) causal or non-causal? Explain your answer in each case.

c) (5 points) Find the current \( i_o \) which will keep the object at equilibrium (i.e., find the necessary current \( i_o \) so that when the initial velocity of the object is zero and the current applied is \( i(t) = i_o \), the object does not move).

d) (8 points) Let \( i(t) = u(t) + i_o \), where \( u \) is considered as the new input and \( i_o \) is as determined above. Write the equation of motion of the system, where \( u \) is the input and \( y \) is the output, and answer part b for this new system.

e) (8 points) Choose a state vector for this new system (with input \( u \) and output \( y \)) and write its state equations.

f) (8 points) Find the transfer function of the system from \( u \) to \( y \).
2. Consider the system described in Question 1. Suppose that the position $y$ of the object is measured by a camera which sends this measurement (let's denote it by $y_m$) to a microprocessor through a digital connection. The microprocessor runs the following code at each sampling interval:

```c
READ Y, R
E = R - Y;
S = S + E;
D = E - EOLD;
U = K1 * E + K2 * S + K3 * D;
EOLD = E;
I = U + IO;
WRITE I
```

where $S$ is a variable, which was initially set to zero, $K1$, $K2$, and $K3$ are positive constants, which were set initially by the designer, and $IO$ is the current $i_o$ that keeps the system at equilibrium (which was calculated in part c of Question 1). READ $Y$, $R$ instruction reads the output $y_m$ of the camera and a reference value $r$, entered by an operator, and assigns them respectively to $Y$ and $R$. WRITE $I$ instruction sends the calculated value of $I$ to a current drive, which applies this current to the electromagnet.

a) (3 points) Does this describe a control system? Why?
b) (5 points) Identify the parts (“plant”, “actuator(s)”, “controller”, “sensor(s)”) of this control system.
c) (5 points) Also identify the “plant input(s)”, “plant output(s)”, “control signal(s)”, “measurement(s)”, “reference(s)”, and “disturbance(s)”, if any.
d) (3 points) Is this an “open-loop” or a “closed-loop” control system? Why?
e) (10 points) Describe the control action. If you are asked to name this control action by choosing one of the names given below, which one would you choose? Why?

i) “bang-bang” or “on/off” (with or without hystereses?)  
ii) proportional  
iii) integral  
iv) derivative  
v) a combination (if so, which?) of above

3. Consider the system described by the following state equations:

$$
\dot{x}(t) = \begin{bmatrix}
-1 & 1 \\
-1 & -1 \\
\end{bmatrix} x(t) + \begin{bmatrix}
0 \\
1 \\
\end{bmatrix} u(t)
$$

$$
y(t) = \begin{bmatrix}
1 & 1 \\
1 & 0 \\
\end{bmatrix} x(t) + \begin{bmatrix}
0 \\
1 \\
\end{bmatrix} u(t)
$$

a) (10 points) Let $x(1) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $u(t) = e^{-t}$ for $t \geq 1$. Find $x(t)$ and $y(t)$ for $t \geq 1$.
b) (10 points) Find the transfer function matrix of the system.
c) (10 points) Determine the minimum time constant, $\tau_{\text{min}}$, of the system. Then, find the state-space equations of the discrete-time equivalent of the system, where the sampling period (or integration step size) is $T = \frac{\tau_{\text{min}}}{10}$ (if you can not find $\tau_{\text{min}}$ take $T = 0.05$).
SOLUTIONS

1. a) By the Newton’s second law:
\[ M \ddot{y}(t) = -Mg + ki(t) - \beta \dot{y}(t) \]  
(1)

b) (i) The system is a dynamic system, since its motion is described by a differential equation.
(ii) It is non-linear, since the differential equation describing the system’s motion is not linear (due to the term \(-Mg\)).
(iii) It is time-invariant, since the differential equation describing the system’s motion has constant coefficients.
(iv) It is causal, since it is not possible for the future inputs to effect the present output.

c) By letting \( \ddot{y}(t) = \dot{y}(t) = 0 \) and \( i(t) = i_o \) in (1), we obtain \( 0 = -Mg + ki_o \). Thus \( i_o = \frac{Mg}{k} \).

d) By letting \( i(t) = u(t) + i_o = u(t) + \frac{Mg}{k} \) in (1), we obtain:
\[ M \ddot{y}(t) = ku(t) - \beta \dot{y}(t) \]  
(2)

The new system is dynamic, time-invariant, and causal, as before. But now it is linear, since the new differential equation is linear in \( u, y \), and their derivatives.

e) A state vector can be chosen as \( x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix} \). Then, \( \dot{x}_1(t) = \dot{y}(t) = x_2(t) \), by using (2): \( \dot{x}_2(t) = \ddot{y}(t) = \frac{ku(t) - \beta x_2(t)}{M} \), and \( y(t) = x_1(t) \). Thus, the state equations can be written as:
\[ \dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & -\beta/M \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1/M \end{bmatrix} u(t) \]  
(3)
\[ y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) \]  
(4)

f) By taking the Laplace transform of both sides of (2), by assuming zero initial conditions (i.e., \( y(0) = \dot{y}(0) = 0 \)), and by rearranging terms, we obtain: \( (Ms^2 + \beta s) \hat{y}(s) = k \hat{u}(s) \). Thus, the transfer function of the system is: \( G(s) = \frac{k}{Ms^2 + \beta s} \).

Alternatively, the same answer can be obtained as \( G(s) = C(sI - A)^{-1}B \), where \( A, B, \) and \( C \) are the matrices appearing in (3)–(4) and \( I \) is the identity matrix of appropriate dimensions.

2. a) The system described is a control system, since a variable (the position, \( y \), of the object) is controlled.

b) The system described in Question 1 is the plant, since one of its variables (\( y \)) is to be controlled. The current drive, which supplies the control input (i.e., the current \( i \)) for the plant, is the actuator. The controller is the microprocessor, since it calculates the necessary control signal for the actuator. The camera is the sensor, since it measures the plant output \( y \).

c) The plant input is the current \( i \) applied to the electromagnet. The plant output is the position \( y \) of the object. The control signal is the current value \( i \), send from the microprocessor to the current drive. The output of the camera, \( y_m \), is the measurement. The reference signal is the reference value \( r \), entered by the operator. No disturbance is specified in the problem (any force, that is not modelled and acts against the movement of the object (e.g.,
air resistance) or that alters the effect of the electromagnet would be a disturbance - note that the viscous friction inside the pipe is not a disturbance, since it is modelled).

d) The system is a closed-loop control system, since the control action depends on the measured value of the plant output.

e) The control action can be described as follows: At each sampling cycle, the measured value of the plant output (y) and of the current reference (r) are obtained first (by the read y, r instruction). Then, y is subtracted from the reference value, r, to find the error, E (E = R - Y). This value is then added to the previous sum, s, of the error (i.e., the error is integrated) by the s = s + E instruction. Then, the difference between the present and the previous error (i.e., the derivative of the error) is calculated by the d = e instruction. Then, the weighted sum of the error plus its integral, plus its derivative is calculated by the u = k1 * e + k2 * s + k3 * d instruction. The value of \( i_o \) is then added to this sum to obtain the control signal. Since the control signal is a weighted sum of the error, its integral, and its derivative, best name for this controller would be a proportional-plus-integral-plus-derivative (PID) controller.

3. Let \( A := \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} \), \( B := \begin{bmatrix} 0 \\ 1 \end{bmatrix} \), \( C := \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \), and \( D := \begin{bmatrix} 0 \\ 1 \end{bmatrix} \). Also let \( L \) denote the Laplace transform.

a) \( e^{At} = L^{-1}[sI - A]^{-1} = \begin{bmatrix} e^{-t} \cos(t) & e^{-t} \sin(t) \\ -e^{-t} \sin(t) & e^{-t} \cos(t) \end{bmatrix} \).

\[ x(t) = e^{A(t-1)}x(1) + \int_1^t e^{A(t-\tau)}Bu(\tau)d\tau = \begin{bmatrix} e^{-t+1} \cos(t-1) - \sin(t-1) \\ -e^{-t+1} \sin(t-1) + \cos(t-1) \end{bmatrix}, \quad t \geq 1. \]

\[ y(t) = Cx(t) + Du(t) = \begin{bmatrix} e^{-t} (1 - \cos(t-1) + (1 - 2e^t) \sin(t-1)) \\ e^{-t} (2 + (e^t - 1) \cos(t-1) - e^t \sin(t-1)) \end{bmatrix}, \quad t \geq 1. \]

c) Eigenvalues of \( A \) are the roots of \( \det(sI - A) = s^2 + 2s + 2 = 0 \), which are \( \lambda_{1,2} = -1 \pm j \).

Thus, \( \tau_{\min} = \frac{1}{\max_{i \in \{|\lambda_i|\}}} = \frac{1}{\sqrt{2}} \). Thus we let \( T = \frac{\tau_{\min}}{10} = \frac{1}{10\sqrt{2}} = 0.07 \), and the discrete-time equivalent of the system is described as

\[ x_d(k+1) = A_dx_d(k) + B_du_d(k) \]

\[ y_d(k) = Cx_d(k) + Du_d(k) \]

where \( A_d = e^{AT} = \begin{bmatrix} e^{-T} \cos(T) & e^{-T} \sin(T) \\ -e^{-T} \sin(T) & e^{-T} \cos(T) \end{bmatrix} = \begin{bmatrix} 0.9294 & 0.0658 \\ -0.0658 & 0.9294 \end{bmatrix} \) and

\[ B_d = \int_0^T e^{A\tau}Bd\tau = \begin{bmatrix} \frac{1}{2} - \frac{1}{4} e^{-T} (\cos(T) + \sin(T)) \\ \frac{1}{4} e^{-T} (\cos(T) - \sin(T)) + \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 0.0024 \\ 0.0682 \end{bmatrix}. \]

Note that \( T \) must be in radians when you calculate \( \sin(T) \) and \( \cos(T) \). If you use degrees, you will get a wrong answer.